General Vector Space (3A)

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1/21/15

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Vector Space

V: non-empty set of objects

defined operations: addition $\mathbf{u} + \mathbf{v}$

scalar multiplication $k\,\mathbf{u}$

if the following axioms are satisfied

for all object **u**, **v**, **w** and all scalar *k*, *m*

⇒ V: vector space

objects in V: vectors

1. if **u** and **v** are objects in V, then **u** + **v** is in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if *k* is any scalar and **u** is objects in V, then *k***u** is in V
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

Test for a Vector Space

1. Identify the set V of objects
2. Identify the addition and scalar multiplication on V
3. Verify $\mathbf{u} + \mathbf{v}$ is in V and $k\mathbf{u}$ is in V

   closure under **addition** and **scalar multiplication**
4. Confirm other axioms.

1. if $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
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6. if k is any scalar and $\mathbf{u}$ is objects in V, then $k\mathbf{u}$ is in V
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8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

Subspace

a subset W of a vector space V

If the subset W is itself a vector space ⇒ the subset W is a **subspace** of V

1. if **u** and **v** are objects in W, then **u** + **v** is in W
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
5. **u** + (–**u**) = (–**u**) + (**u**) = **0**
6. if *k* is any scalar and **u** is objects in W, then *k***u** is in W
7. *k*(**u** + **v**) = *k***u** + *k***v**
8. (*k* + *m*)**u** = *k***u** + *m***u**
9. *k*(*m***u**) = (*km*)**u**
10. *1*(**u**) = **u**

Subspace Test (1)

a subset W of a vector space V

If the subset W is itself a vector space ➡ the subset W is a **subspace** of V

axioms not inherited by W

1. if **u** and **v** are objects in W, then **u** + **v** is in W
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
5. **u** + (–**u**) = (–**u**) + (**u**) = **0**
6. if *k* is any scalar and **u** is objects in W, then *k***u** is in W
7. *k*(**u** + **v**) = *k***u** + *k***v**
8. (*k* + *m*)**u** = *k***u** + *m***u**
9. *k*(*m***u**) = (*km*)**u**
10. *1*(**u**) = **u**

Subspace Test (2)

a subset W of a vector space V

if $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$ if k: a scalar, $\mathbf{u} \in W$, then $k\mathbf{u} \in W$	⬌	the subset W is a **subspace** of V

1. if **u** and **v** are objects in W, then **u** + **v** is in W
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ (zero vector)
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + (\mathbf{u}) = \mathbf{0}$
6. if k is any scalar and **u** is objects in W, then $k\mathbf{u}$ is in W
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
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10. $1(\mathbf{u}) = \mathbf{u}$

Linear Combination : Subspaces

$S = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_r\}$

a nonempty set of a vector space V

S may not be a vector space subspace of V

$W = \{\mathbf{w} \mid \mathbf{w} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_r\mathbf{w}_r\}$

but all linear combination of the vectors in S is a subspace of V

the set W of all possible linear combination of the vectors in S

$\mathbf{w} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_r\mathbf{w}_r$

⇒ a subspace of V

Closure : Subspaces

$\mathbf{u} \in W, \quad \mathbf{v} \in W$ ⇒ $\mathbf{u} + \mathbf{v} \in W, \quad k\mathbf{u} \in W$

$$\begin{cases} \mathbf{u} = c_1\mathbf{w_1} + c_2\mathbf{w_2} + \cdots + c_r\mathbf{w_r} \\ \mathbf{v} = k_1\mathbf{w_1} + k_2\mathbf{w_2} + \cdots + k_r\mathbf{w_r} \end{cases}$$

⇒

$$\begin{cases} \mathbf{u} + \mathbf{v} : \text{ a linear combination} \\ k\mathbf{u} : \text{ a linear combination} \end{cases}$$

closure under addition

$$\mathbf{u} + \mathbf{v} = (c_1 + k_1)\mathbf{w_1} + (c_2 + k_2)\mathbf{w_2} + \cdots + (c_r + k_r)\mathbf{w_r}$$

closure under scalar multiplication

$$k\mathbf{u} = (k c_1)\mathbf{w_1} + (k c_2)\mathbf{w_2} + \cdots + (k c_r)\mathbf{w_r}$$

The Smallest Subspaces

the set W is the smallest subspace of V that contains *all of the vectors* in S
any other subspace that contains *all of the vectors* in S, contains W

vector space
- closure under addition
- closure under scalar multiplication

the subspace W' contains all the vectors in S
$S = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_r\}$

→ the subspace W' contains all possible linear combination of the vectors in S

$S \subset W'$ → $W \subset W'$

Spanning Set

$$S = \{\boldsymbol{w}_{1,}\boldsymbol{w}_{2,} \cdots ,\boldsymbol{w}_{r}\}$$

a nonempty set of a vector space V

$$W = \{\boldsymbol{w} \mid \boldsymbol{w} = c_1\boldsymbol{w}_1 + c_2\boldsymbol{w}_2 + \cdots + c_r\boldsymbol{w}_r\}$$

all linear combination of the vectors in S
is a subspace of V

$$span(S) = span\{\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots , \boldsymbol{w}_k\}$$

Spanning Set : not unique

$S_1 = \{\mathbf{v}_{1,}\ \mathbf{v}_{2,}\ \cdots\ ,\mathbf{v}_r\}$ a nonempty set of a vector space V

$S_2 = \{\mathbf{w}_{1,}\mathbf{w}_{2,}\ \cdots\ ,\mathbf{w}_k\}$ a nonempty set of a vector space V

$$span\{\mathbf{v}_{1,}\ \mathbf{v}_{2,}\ \cdots\ ,\mathbf{v}_r\} = span\{\mathbf{w}_{1,}\mathbf{w}_{2,}\ \cdots\ ,\mathbf{w}_k\} \Leftrightarrow$$

each vector in S_1 is a linear combination of the vectors in S_2

each vector in S_2 is a linear combination of the vectors in S_1

Containment : Subspaces

$S = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_r\}$ ➡ S may not be a subspace of V

$W = \{\mathbf{w} \mid \mathbf{w} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_r\mathbf{w}_r\}$ ➡ W is a subspace of V

$W' = \{\mathbf{w} \mid \mathbf{w} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_q\mathbf{w}_q\}$

If W' is a subspace of V and contains all the vectors in S

$q > r$ ➡ W' contains W

$span(W') \geq span(W)$

$q = r$ ➡ W' contains W

$span(W') = span(W)$

$q < r$ the vectors in S are linearly dependent ➡ W' contains W

$span(W') = span(W)$

Building Subspaces

if W_1, W_2, … , W_n are subspaces of a vector space of V → the intersection of these subspaces are also a subspace of V

$S = \{w_1, w_2, \cdots, w_r\}$ a nonempty set of a vector space V

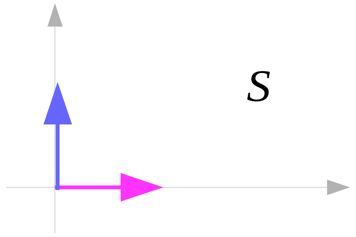
the set W of all possible linear combination of the vectors in S

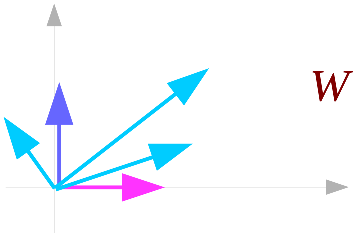
$$w = c_1 w_1 + c_2 w_2 + \cdots + c_r w_r$$

→ a subspace of V

the set W is the smallest subspace of V that contains *all of the vectors* in S

any other subspace that contains *all of the vectors* in S contains W



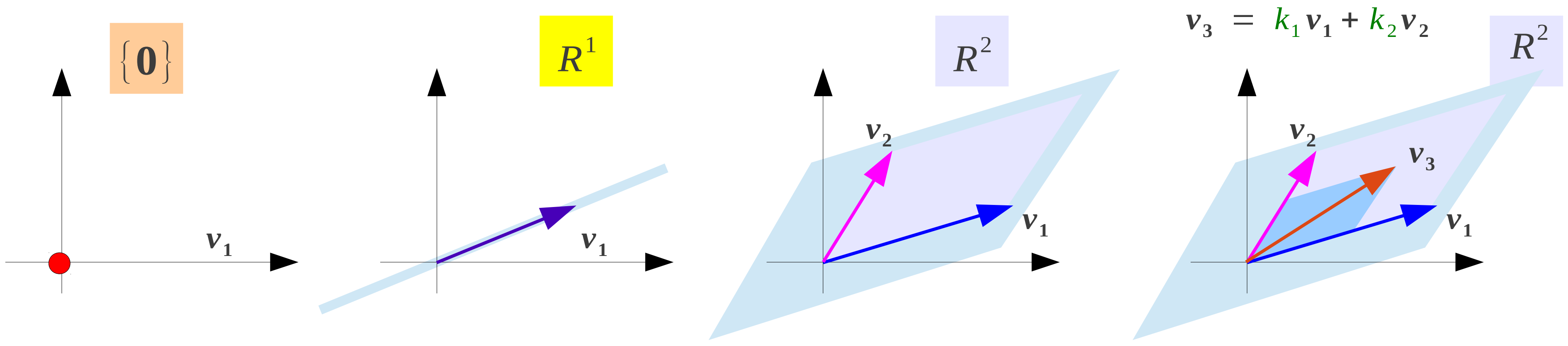


Subspace Example (1)

In vector space R^2

any one vector	(linearly indep.)	**spans**	R^1	line through **0**
any two non-collinear vectors	(linearly indep.)	**spans**	R^2	plane
any three or more vectors	(linearly dep.)	**spans**	R^2	plane

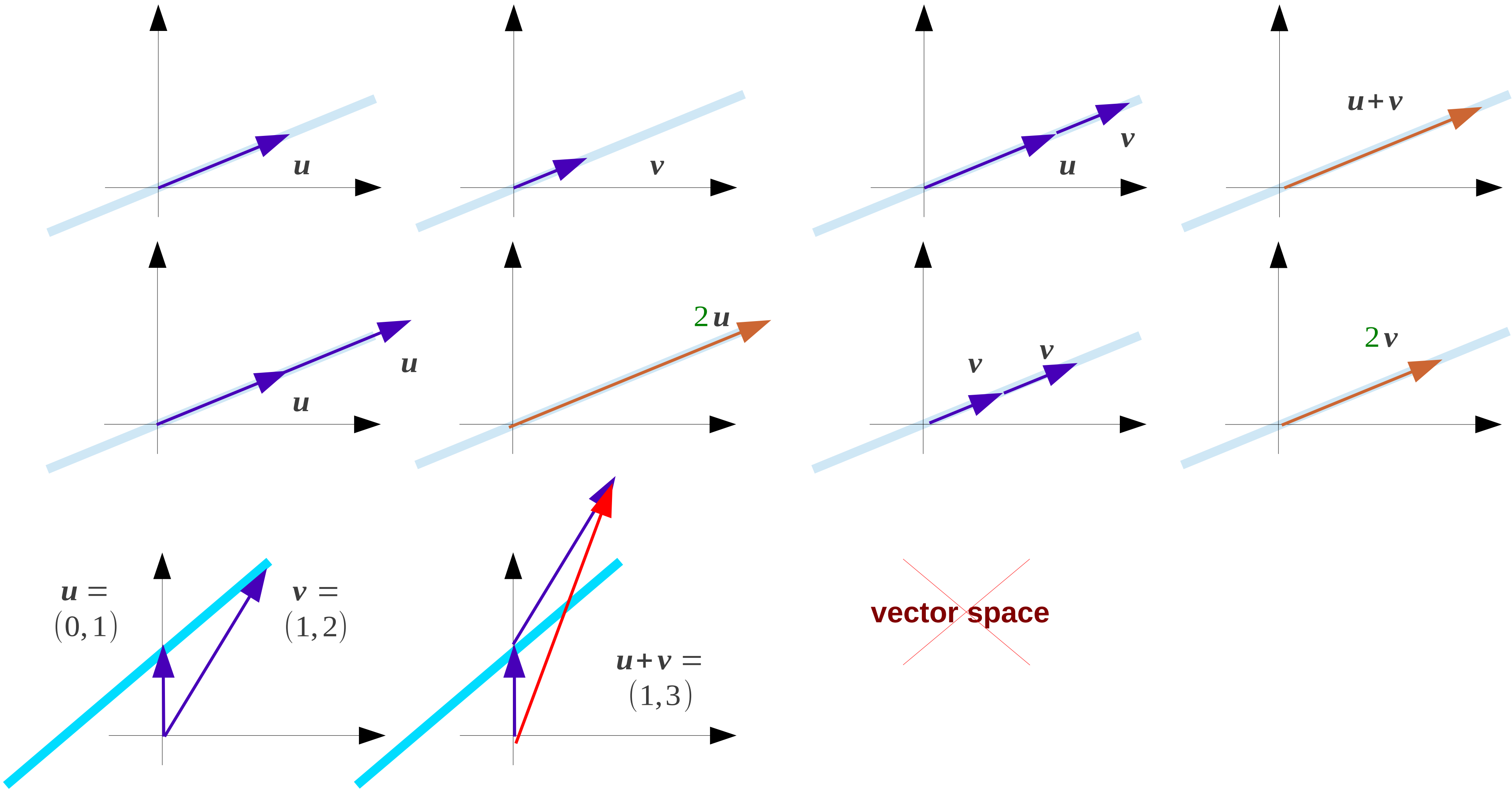
Subspaces of R^2



Subspace Example (2)

In vector space R^2

any one vector (linearly indep.) **spans** R^1 line through **0**



Subspace Example (3)

In vector space R^3

any one vector	(linearly indep.)	**spans**	R^1	line through $\mathbf{0}$
any two non-collinear vectors	(linearly indep.)	**spans**	R^2	plane through $\mathbf{0}$
any three vectors non-collinear, non-coplanar	(linearly indep.)	**spans**	R^3	3-dim space
any four or more vectors	(linearly dep.)	**spans**	R^3	3-dim space

Subspaces of R^3

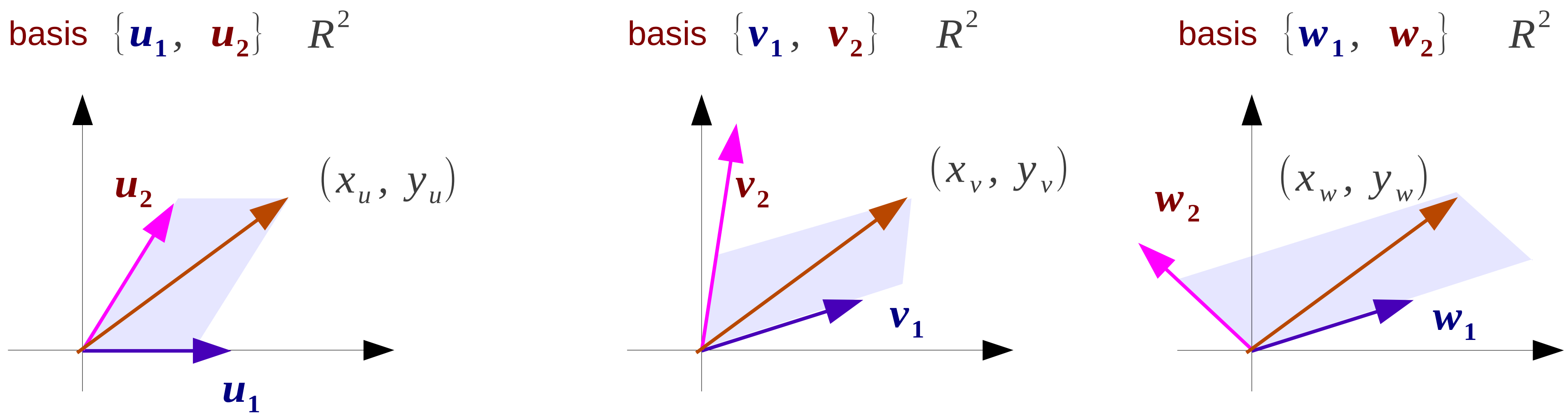
$\{\mathbf{0}\}$	R^1	R^2	R^3
	line through $\mathbf{0}$	plane through $\mathbf{0}$	3-dim space

Dimension

In a finite-dimensional vector space R^n ~~R^∞~~

all bases ➡ the same number of vectors n

many bases but the same number of basis vectors



The dimension of a finite-dimensional vector space V

dim(V) ⟺ the number of vectors in a basis

Dimension of a Basis (1)

In vector space R^2

	any one vector	(linearly indep.)	~~spans~~	~~R^2~~	line through **0**
basis	any two non-collinear vectors	(linearly indep.)	**spans**	R^2	plane
	any three or more vectors	~~(linearly indep.)~~	**spans**	R^2	plane

In vector space R^3

	any one vector	(linearly indep.)	~~spans~~	~~R^3~~	line through **0**
	any two non-collinear vectors	(linearly indep.)	~~spans~~	~~R^3~~	plane through **0**
basis	any three vectors non-collinear, non-coplanar	(linearly indep.)	**spans**	R^3	3-dim space
	any four or more vectors	~~(linearly indep.)~~	**spans**	R^3	3-dim space

Dimension of a Basis (2)

In vector space R^n

	any n-1 vectors	(linearly indep.)?	~~spans~~	~~R^n~~	line through **0**
basis	n vectors of a basis	(linearly indep.)	**spans**	R^n	plane
	any n+1 vectors	~~(linearly indep.)~~	**spans**?	R^n	plane

a finite-dimensional vector space V

a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$

- a set of more than n vectors ➡ ~~(linearly indep.)~~
- a set of less than n vectors ➡ ~~**spans** V~~

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ non-empty finite set of vectors in V

S is a basis ⟺
- S linearly independent
- S spans V

Basis Test

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ non-empty finite set of vectors in V

S is a basis ⇔ { S linearly independent; S spans V }

V an **n**-dimensional vector space

$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ a set of **n** vectors in V

S linearly independent ⇒ S is a basis

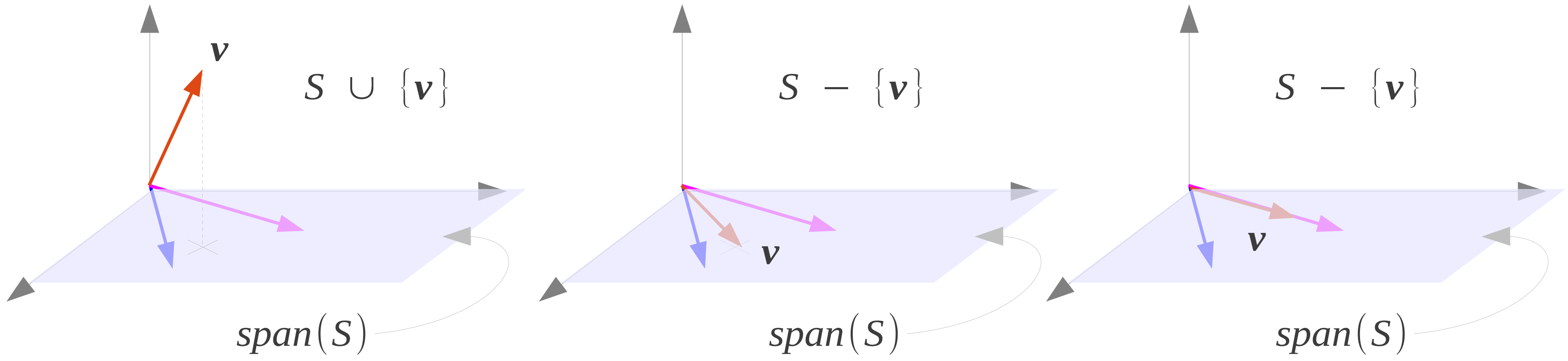
S spans V ⇒ S is a basis

Plus / Minus Theorem

S a nonempty set of vectors in a vector space V

$\begin{cases} S & \text{: linear independent} \\ \boldsymbol{v} & \text{a vector in V but outside of span(S)} \end{cases}$ ➡ $S \cup \{\boldsymbol{v}\}$: linear independent

$\begin{cases} \boldsymbol{v},\ \boldsymbol{u_i} \in S & \text{linear combination} \\ \boldsymbol{v} = k_1 \boldsymbol{u_1} + k_2 \boldsymbol{u_2} + \cdots + k_n \boldsymbol{u_n} \end{cases}$ ➡ $span(S) = span(S - \{\boldsymbol{v}\})$



Finding a Basis

S a nonempty set of vectors in a vector space V

$\begin{cases} S & \text{: linear independent} \\ \mathbf{v} & \text{a vector in V but outside of span(S)} \end{cases} \Rightarrow S \cup \{\mathbf{v}\}$: linear independent

if S is a *linearly independent* set that is *not already a basis* for V,
then S can be *enlarged* to a basis for V
by ***inserting*** appropriate vectors into S

$\begin{cases} \mathbf{v}, \ \mathbf{u_i} \in S & \text{linear combination} \\ \mathbf{v} = k_1 \mathbf{u_1} + k_2 \mathbf{u_2} + \cdots + k_n \mathbf{u_n} \end{cases} \Rightarrow span(S) = span(S - \{\mathbf{v}\})$

if S *spans* V but is *not a basis* for V,
then S can be *reduced* to a basis for V
by ***removing*** appropriate vectors from S

Vectors in a Vector Space

S a nonempty set of vectors in a vector space V

if S is a *linearly independent* set that is *not already a basis* for V,
then S can be *enlarged* to a basis for V
by ***inserting*** appropriate vectors into S

Every *linearly independent* set in a subspace is
either a **basis** for that subspace
or can be **extended to a basis** for it

if S *spans* V but is *not a basis* for V,
then S can be *reduced* to a basis for V
by ***removing*** appropriate vectors from S

Every *spanning set* for a subspace is
either a **basis** for that subspace
or has a **basis as a subset**

Dimension of a Subspace

W a subspace of a finite-dimensional vector space V

W is *finite-dimensional*

dim(W) ≤ dim(V)

W = V ⟺ dim(W) = dim(V)

Vector Space Examples

$\{\mathbf{0}\}$

R^n

M_{mn} mxn matrix

$F(-\infty, +\infty)$ real-valued functions in the interval $(-\infty, +\infty)$

$C(-\infty, +\infty)$ real-valued continuous functions in the interval $(-\infty, +\infty)$

$C^1(-\infty, +\infty)$ real-valued continuously differentiable functions in $(-\infty, +\infty)$

P_∞ $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

the solution space $\mathbf{A}\mathbf{x} = \mathbf{0}$ in n unknowns R^n

Real-Valued Functions (1)

V the set of real-valued functions

defined at every x in $(-\infty, +\infty)$

$$\boldsymbol{u} = u(x) \qquad \boldsymbol{u}+\boldsymbol{v} = u(x)+v(x)$$

$$\boldsymbol{v} = v(x) \qquad k\boldsymbol{u} = k\,u(x)$$

1. if **u** and **v** are objects in V, then **u** + **v** is in V
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
5. **u** + (–**u**) = (–**u**) + (**u**) = **0**
6. if *k* is any scalar and **u** is objects in V, then *k***u** is in V
7. *k*(**u** + **v**) = *k***u** + *k***v**
8. (*k* + *m*)**u** = *k***u** + *m***u**
9. *k*(*m***u**) = (*km*)**u**
10. *1*(**u**) = **u**

Real-Valued Functions (2)

V the set of real-valued functions $\{\sin(x),\ \sin(2x),\ \sin(3x),\ \cdots\}$

defined at every x in $[0,\ 2\pi]$

$$\boldsymbol{u_1} = \sin(x)$$
$$\boldsymbol{u_2} = \sin(2x)$$
$$\boldsymbol{u_3} = \sin(3x)$$
$$\cdots$$

$$\boldsymbol{u_m} + \boldsymbol{v_n} = \sin(mx) + \sin(nx)$$
$$k\,\boldsymbol{u_m} = k\sin(mx)$$

1. if **u** and **v** are objects in V, then **u** + **v** is in V
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
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9. *k*(*m***u**) = (*km*)**u**
10. *1*(**u**) = **u**

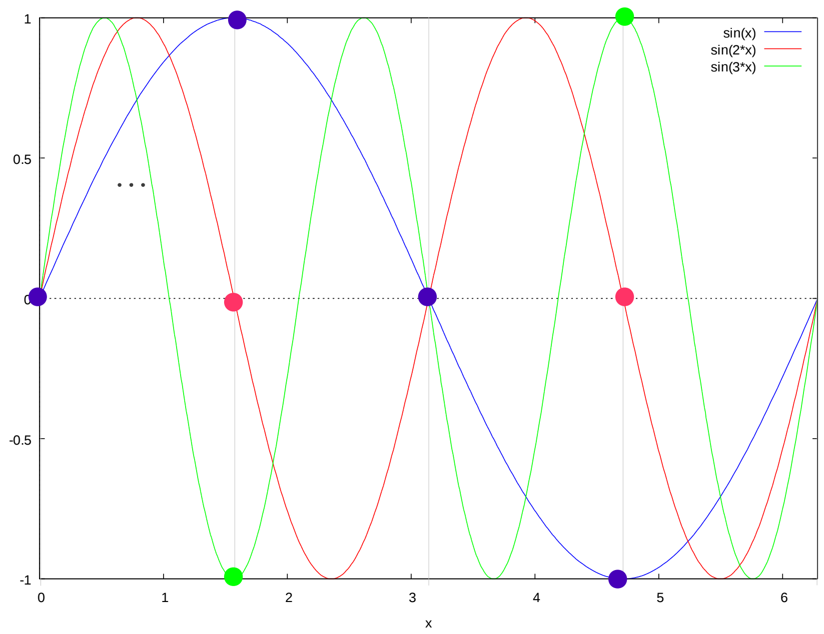
V basis R^{∞}

linear independent

Real-Valued Functions (3)

$$\boldsymbol{u}_1 = [\sin(0),\ \sin(\pi/2),\ \sin(\pi),\ \sin(3\pi/2)]$$
$$= [0.00000\ \ 0.70711\ \ 1.00000\ \ 0.70711]$$
$$\boldsymbol{u}_2 = [\sin(2\cdot 0),\ \sin(2\cdot\pi/2),\ \sin(2\cdot\pi),\ \sin(2\cdot 3\pi/2)]$$
$$= [0.00000\ \ 1.00000\ \ 0.00000\ \ -1.00000]$$
$$\boldsymbol{u}_3 = [\sin(3\cdot 0),\ \sin(3\cdot\pi/2),\ \sin(3\cdot\pi),\ \sin(3\cdot 3\pi/2)]$$
$$= [0.00000\ \ 1.00000\ \ 0.00000\ \ -1.00000]$$

4-tuple vectors



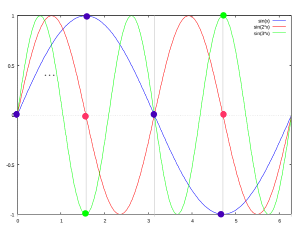
8-tuple vectors
12-tuple vectors
1024-tuple vectors
infinity-tuple vectors

R^{∞}

Real-Valued Functions (4)

$\{\sin(x),\ \sin(2x),\ \sin(3x),\ \cdots\}$ a basis

$+1$



-1

0 2π

span

$+\infty$

$-\infty$

0 2π

8-tuple vectors
12-tuple vectors
1024-tuple vectors
infinity-tuple vectors

R^{∞}

$dim(R^{\infty}) = \infty$

$m \neq n$

$\sin(mx) \neq \sin(nx)$

linearly independent

References

[1] http://en.wikipedia.org/
[2] Anton, et al., Elementary Linear Algebra, 10th ed, Wiley, 2011
[3] Anton, et al., Contemporary Linear Algebra,